Research Master's Thesis Proposal: Energy Conserving Exponential Integrators for Poisson Systems

Motivation and Current State of the Art For the integration of stiff systems of ODEs, exponential integrators offer an alternative to the use of fully implicit or semi-implicit methods. In particular, they can allow longer time steps and have better behavior for the high-frequency modes ([3]). Additionally, many interesting systems of ODEs have conserved quantities, and it can be useful to have integrators capable of also conserving these quantities. A system of ODEs with a conserved quantity $\mathcal{H}$ can always be written as:

$$\frac{\partial \vec{x}}{\partial t} = \mathbb{J}(\vec{x}) \frac{\delta \mathcal{H}}{\delta \vec{x}}$$

where $\mathbb{J}(\vec{x})$ is a skew-symmetric matrix that depends on $\vec{x}$ and $\frac{\delta \mathcal{H}}{\delta \vec{x}}$ is the functional derivative of the conserved quantity $\mathcal{H}$. This is known as a Poisson system, and there is a class of implicit time integrators that is capable of conserving $\mathcal{H}$ for arbitrary $\mathbb{J}$ [1]. Independently, recent work [2] has produced an exponential integrator that can conserve energy for constant $\mathbb{J}$, but not the more general case of $\mathbb{J}$ that depends on $\vec{x}$. This is the focus of this thesis.

Objectives

1. Develop an energy conserving exponential integrator for Poisson systems with arbitrary $\mathbb{J}$, by combining the ideas in [1] and [2].
2. Test this integrator on simple Poisson systems, and compare to existing state of the art implicit and exponential integrators in the literature. Particular attention should be paid to the behavior for long time steps.

If time permits, the integrator will be applied to geophysical fluid dynamical systems (such as the shallow water equations and the hydrostatic Boussinesq equations) and compared to an existing implicit conserving scheme. Many systems of interest in geophysical fluid dynamics are stiff, and furthermore accurate resolution of the behavior of the fast modes is important. For example, in ocean models split-explicit time integration is used to resolve the fast barotropic modes at a much higher temporal resolution than the slow baroclinic modes. Exponential integrators might offer an alternative to this splitting.

Required Skills Applied Mathematics (Calculus, Linear Algebra, ODE Integrators, Numerical Methods), Scientific Computing (Fortran/C/Python preferred), English

Location AIRSEA Team, Laboratoire Jean Kuntzmann, Grenoble

Supervisors

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Bibliography

1. Cohen, D. & Hairer, E. *Linear energy-preserving integrators for Poisson systems.* Bit Numer Math (2011) 51: 91.
2. Shen.X & Leok, M. *Geometric Exponential Integrators*, arXiv:1703.00929 [math.NA]
3. Gaudreault S. & Pudykiewicz J. *An efficient exponential time integration method for the numerical solution of the shallow water equations on the sphere.* Journal of Computational Physics (2016).